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NON-LINEAR OSCILLATIONS OF REVERSIBLE SYSTEMS†

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A reversible system with a small parameter μ is considered. When μ = 0 the generating system has a periodic motion, symmetric to a fixed set of the system automorphism. It is shown that this periodic motion is continued with respect to a small parameter in the Poincaré-unisolated case when certain conditions are satisfied only on the generating system. Symmetric periodic solutions are constructed both for a non-resonant and for a resonant system. In the plane unrestricted three-body problem the small parameter is chosen to be the quantity characterizing the interaction between two bodies chosen from the three. It is shown that in this problem there are solutions in which the body moves along curves close to circles. The problem of the applicability of the result to a sun–earth–moon type is discussed.

1. THE LYAPUNOV–POINCARÉ METHOD IN REVERSIBLE SYSTEMS

Consider the reversible system

$$\mathbf{u}^{\cdot} = \mathbf{U}(\mathbf{u}, \mathbf{v}) + \mu\mathbf{U}_1(\mu, \mathbf{u}, \mathbf{v}, t)$$
$$\mathbf{v}^{\cdot} = \mathbf{V}(\mathbf{u}, \mathbf{v}) + \mu\mathbf{V}_1(\mu, \mathbf{u}, \mathbf{v}, t), \quad \mathbf{u} \in \mathbf{R}^l, \mathbf{v} \in \mathbf{R}^n \quad (l \geq n) \tag{1.1}$$

where μ is a small parameter, $\mathbf{U}_1$ and $\mathbf{V}_1$ are 2π-periodic functions of t or are independent of t, while $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = 0\}$ is a fixed automorphism set. Suppose that, when $\mu = 0$, system (1.1) admits of $2\pi k$-periodic ($k \in \mathbf{N}$) motion $\mathbf{u} = \varphi(t)$, $\mathbf{v} = \psi(t)$, which intersects the set M at the instant of time $t = 0$ (mod 2π). We make the replacement

$$\mathbf{u} = \varphi(t) + \mathbf{p}, \quad \mathbf{v} = \psi(t) + \mathbf{q}$$

Then the equations for **p** and **q**

$$\mathbf{p}^{\cdot} = \mathbf{A}(t)\mathbf{p} + \mathbf{B}(t)\mathbf{q} + \mathbf{P}(\mathbf{p}, \mathbf{q}, t) + \mu\mathbf{U}_1(\mu, \varphi(t) + \mathbf{p}, \psi(t) + \mathbf{q}, t)$$
$$\mathbf{q}^{\cdot} = \mathbf{C}(t)\mathbf{p} + \mathbf{D}(t)\mathbf{q} + \mathbf{Q}(\mathbf{p}, \mathbf{q}, t) + \mu\mathbf{V}_1(\mu, \varphi(t) + \mathbf{p}, \psi(t) + \mathbf{q}, t) \tag{1.2}$$

where we have denoted terms higher than the first order of **p** and **q** by **P** and **Q**, are reversible [1], while the fixed automorphism set of system (1.2) coincides with the hyperplane $\mathbf{q} = 0$. The right-hand sides of Eqs (1.2) are $2\pi k$-periodic functions of t.

When $\mu = 0$ system (1.2) has the solution $\mathbf{p} = \mathbf{0}, \mathbf{q} = \mathbf{0}$. We will formulate the problem of the existence of $2\pi k^*$ periodic solutions of system (1.2) (where k^* is a multiple of k) when $\mu \neq 0$.

Suppose κ_s are characteristic exponents of the linear system

$$\mathbf{p}^{\cdot} = \mathbf{A}(t)\mathbf{p} + \mathbf{B}(t)\mathbf{q}, \quad \mathbf{q}^{\cdot} = \mathbf{C}(t)\mathbf{p} + \mathbf{D}(t)\mathbf{q} \tag{1.3}$$

Then when $n = l$ and $\kappa_s \neq i\nu/k^*$ ($\nu \in \mathbf{Z}$) Poincaré's theorem [2] solves the problem.

The case of zero κ_s with simple elementary divisors was investigated in [3]. The solution obtained requires the construction of a system of independent periodic solutions corresponding to the multiple zero root of the system conjugate to (1.3), and the inclusion for the analysis of the terms in (1.2) which depend on the small parameter.

It turns out that, for the reversible system considered, the problem has a solution that is more complete and convenient for applications.

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Lemma 1. The linear reversible system (1.3) has at least $l-n$ zero characteristic exponents and simple elementary divisors correspond to them.

Proof. In view of the reversibility of system (1.3), in addition to the solution $\mathbf{p} = \varphi^*(t)$, $\mathbf{q} = \psi^*(t)$, we also have the solutions

$$\mathbf{p} = \varphi^*(t) + \varphi^*(-t), \quad \mathbf{q} = \psi^*(t) - \psi^*(-t) \tag{1.4}$$

$$\mathbf{p} = \varphi^*(t) - \varphi^*(-t), \quad \mathbf{q} = \psi^*(t) + \psi^*(-t) \tag{1.5}$$

Since in (1.4) we have $\mathbf{q}(0) = \mathbf{0}$ and in (1.5) we have $\mathbf{p}(0) = \mathbf{0}$, we can always construct a fundamental system of solutions, in each of the solutions of which $\mathbf{p}$ is an even (odd) function of t while $\mathbf{q}$ is an odd (even) function of t. In view of the uniqueness, the fundamental system with the unit matrix E of the initial conditions has the form

$$\mathrm{S}(t) = \begin{Vmatrix} \mathbf{p}^+(t) & \mathbf{p}^-(t) \\ \mathbf{q}^-(t) & \mathbf{q}^+(t) \end{Vmatrix}, \quad \mathrm{S}(0) = \mathbf{E}$$

where the plus (minus) denotes a matrix of even (odd) functions.

Suppose G is a characteristic matrix. Then $\mathrm{S}(k\pi) = \mathrm{GS}(-k\pi)$, and the characteristic equation $\det(\mathrm{G} - \rho\mathrm{E}) = 0$ is equivalent to the equation $\det(\mathrm{S}(k\pi) - \rho\mathrm{S}(-k\pi)) = 0$. Also, the matrix $\mathrm{S}(k\pi) - \mathrm{S}(-k\pi)$ has at least $l-n$ zero eigenvalues with simple elementary divisors.

Corollaries. 1. System (1.3) can be reduced by means of a non-degenerate linear transformation to the following form

$$\xi^{\cdot} = \mathbf{0}, \quad \eta^{\cdot} = \mathbf{A}^*(t)\eta + \mathbf{B}^*(t)\zeta, \quad \zeta^{\cdot} = \mathbf{C}^*(t)\eta + \mathbf{D}^*(t)\zeta \tag{1.6}$$

($\zeta \in \mathbf{R}^{l-n}$; $\eta, \zeta \in \mathbf{R}^n$) with automorphism $t \to -t$, $(\xi, \eta, \zeta) \to (\xi, \eta, -\zeta)$.

2. The linear system

$$\mathbf{p}^{\cdot} = \mathbf{A}(t)\mathbf{p}, \ \mathbf{A}(-t) = -\mathbf{A}(t), \ \mathbf{A}(t+T) = \mathbf{A}(t) \quad (T \neq 0);\ \mathbf{p} \in \mathbf{R}^l$$

is stable and has l integrals T-periodic with respect to t and linear with respect to $\mathbf{p}$.

Lemma 2. Suppose $\kappa_1, \ldots, \kappa_\alpha, \pm\kappa_{\alpha+1}, \ldots, \pm\kappa_m$ are the remaining characteristic exponents of system (1.3) of multiplicity $\beta_1, \ldots, \beta_m$, respectively, where $\kappa_1 = \ldots = \kappa_\alpha$, $\kappa_{\alpha+1} \neq 0, \ldots, \kappa_m \neq 0$, and each of the exponents κ is written as many times as there are groups of solutions corresponding to it. Then system (1.3) becomes

$$\begin{aligned}
&\xi^{\cdot} = \mathbf{0} \quad (\xi \in \mathbf{R}^{l-n}), \quad \eta^{\cdot}_{1,\nu} = \kappa_\nu \zeta_{1,\nu} \\
&\eta^{\cdot}_{1,s} = 0, \ \zeta^{\cdot}_{1,s} = \eta_{1,s}, \quad \zeta^{\cdot}_{1,\nu} = \kappa_\nu \eta_{1,\nu} \\
&\eta^{\cdot}_{i,s} = \zeta_{i-1,s}, \quad \eta^{\cdot}_{j+1,\nu} = \kappa_\nu \zeta_{j+1,\nu} + \zeta_{j,\nu} \\
&\zeta^{\cdot}_{i,s} = \eta_{i,s}, \quad \zeta^{\cdot}_{j+1,\nu} = \kappa_\nu \eta_{j+1,\nu} + \eta_{j,\nu} \\
&(i = 2, \ldots, \beta_s/2;\ j = 1, \ldots, \beta_\nu - 1;\ s = 1, \ldots, \alpha;\ \nu = \alpha + 1, \ldots, m)
\end{aligned} \tag{1.7}$$

with one of the automorphisms: (1) $t \to -t$, $\xi \to \xi$, $\eta \to \eta$, $\zeta \to -\zeta$; (2) $t \to -t$, $\xi \to \xi$, $\eta \to -\eta$, $\zeta \to \zeta$.

Proof. The truth of the lemma follows from Lyapunov's theorem [4] which states that a linear periodic system can be reduced to a system with constant coefficients and preserves [1] the property of reversibility, and also from the corollary formulated above.

Theorem 1. Suppose system (1.2) in the variables ξ, η, ζ possesses the automorphism $t \to -t$, $(\xi, \eta) \to (\xi, \eta)$, $\zeta \to -\zeta$ and the part of it that is linear in ξ, η, ζ when $\mu = 0$ is identical with (1.7). Then, if $\kappa_\nu \neq \pm iN/k^*$ ($N = 1, \ldots, k^*$), for sufficiently small μ system (1.1) has $l-n+1$ families of $2\pi k$-periodic solutions, parametric from the initial conditions and the parameter μ, symmetric to the fixed set $\mathbf{M}$ of automorphism of the system and which becomes a generating family $\mathbf{u} = \varphi(t)$, $\mathbf{v} = \psi(t)$ when $\mu = 0$.

Corollaries. 1. There are always $2\pi k$-periodic solutions if, in the variables, ξ, η, ζ, the automorphisms has the form $t \to -t$, $(\xi, \eta) \to (\xi, \eta)$, $\zeta \to -\zeta$ and there are no imaginary numbers among the numbers κ_ν.

2. If system (1.3) has not more than $l-n$ zero characteristic exponents, $2\pi k$-periodic solutions exist when $\kappa_\nu \neq \pm iN/k^*$ ($N = 1, \ldots, k^*$).

Note. When determining the form of the automorphism of system (1.2) written in ξ, η, ζ variables, it is useful to use the first integrals of the generating system.

Proof. The existence of $2\pi k^*$-periodic motions can be established from the Heinbockel–Struble theorem [5]. If $\xi(\xi^0, \eta^0, \zeta^0, \mu, t)$, $\eta(\xi^0, \eta^0, \zeta^0, \mu, t)$, $\zeta(\xi^0, \eta^0, \zeta^0\ \mu, t)$ is the solution of system of (1.2) with linear part (1.7) and initial values ξ^0, η^0, ζ^0, then the sufficient conditions for it to be $2\pi k^*$-periodic are

$$\boldsymbol{\zeta}^0 = \mathbf{0}, \quad \boldsymbol{\zeta}(\boldsymbol{\xi}^0, \boldsymbol{\eta}^0, \boldsymbol{\zeta}^0, \mu, \pi k^*) = \mathbf{0} \tag{1.8}$$

If the system of $2n$ functional equations in η^0, ζ^0 obtain is compatible, a periodic solution exists. Then η^0, ζ^0 are found as functions of ξ^0, μ and the $2\pi k$-periodic solutions form $l-n+1$ parametric families.

In the first approximation in ξ^0, η^0, ζ^0 ignoring terms which depend on μ, we can set up system (1.8) by integrating (1.7). In this approximation (1.8) is split into m subsystems corresponding to the characteristic exponents from one group of solutions. Hence, the functional determinant of system (1.8) calculated for $\xi^0 = 0$, $\eta^0 = 0$, $\zeta^0 = 0$, $\mu = 0$ is equal to the product of m functional determinants. As in Poincaré's theorem, for $\kappa_\nu \neq 0$ ($\nu = \alpha + 1, \ldots, m$) the determinant is non-zero when $\kappa_\nu \neq \pm iN/k^*$ ($N = 1, \ldots, k^*$). As regards the zero exponents, the corresponding subsystem has the form

$$c_{21}^{(\nu)} + \ldots = 0, \ldots, \quad c_{2,\beta_s/2}^{(\nu)} + \ldots = 0$$

$$\gamma c_{11}^{(s)} + c_{21}^{(s)} + \ldots = 0, \quad (\gamma = \pi k^*)$$

$$\frac{\gamma^3}{3!} c_{11}^{(s)} + \frac{\gamma^3}{2!} c_{21}^{(s)} + \gamma c_{12}^{(s)} + c_{22}^{(s)} + \ldots = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\gamma^{\beta_s - 1}}{(\beta_s - 1)!} c_{11}^{(s)} + \frac{\gamma^{\beta_s - 1}}{(\beta_s - 2)!} c_{12}^{(s)} + \ldots + \gamma c_{1,\beta_s/2}^{(s)} + c_{2,\beta_s/2}^{(s)} + \ldots = 0$$

(c are the initial values of η^0, ζ^0, while the terms which are omitted in front of the equality sign are non-linear in c or depend on μ) and its functional determinant is obviously non-zero.

Theorem 1 is proved. When using it to investigate specific problems the following formulation is often preferable.

Theorem 2. If among the roots α of the equation $\det \|\mathbf{q}^+(2\pi k) - \alpha \mathbf{E}\| = 0$ there are no roots equal to $\cos(2\pi kN/k^*)$, then, for sufficiently small μ, system (1.2) has $l-n+1$ families of symmetric $2\pi k^*$-periodic solutions, parametric from the initial conditions and the parameter μ, which turn into a generating family $\mathbf{p} = \mathbf{0}$, $\mathbf{q} = \mathbf{0}$ when $\mu = 0$.

Proof. Suppose ρ is the root of the characteristic equation. Then $2\alpha = \rho + \rho^{-1}$ is an eigenvalue of the matrix $\mathbf{S}(2\pi k) + \mathbf{S}(-2\pi k)$. If $l \geqslant n$, then all ρ, with the exception of $l-n$, equal to one, are determined from the equation.

$$\rho^2 - 2\alpha_s \rho + 1 = 0, \quad \det\|\mathbf{q}^+(2\pi k) - \alpha_s \mathbf{E}\| = 0, \quad (s = 1, \ldots, n)$$

which proves the theorem.

Note. To determine the characteristic indices of the linear reversible system it is sufficient to construct only n partial solutions. This is particularly convenient when $n = 1$.

2. A NON-RESONANT OSCILLATING SYSTEM

We will investigate the problem of oscillations in the system

$$\begin{aligned} \eta_s^{\cdot} &= i\omega_s \eta_s + i\eta_s \sum_{j=1}^n C_{sj} \eta_j \bar{\eta}_j + \mu H_s(\mu, \eta, \bar{\eta}, t) \\ \bar{\eta}_s^{\cdot} &= -i\omega_s \bar{\eta}_s - i\bar{\eta}_s \sum_{j=1}^n C_{sj} \eta_j \bar{\eta}_j + \mu \bar{H}_s(\mu, \eta, \bar{\eta}, t) \quad (s = 1, \ldots, n) \end{aligned} \tag{2.1}$$

with right-hand sides that are 2π-periodic in t or independent of t and with automorphism $(t, \eta, \bar{\eta}) \to (-t, \eta, \bar{\eta})$.

Here ω_s, $C_{sj}(\omega_s > 0)$ are real constants and the bar indicates a complex-conjugate quantity.

This problem arises when investigating the small oscillations of a reversible system in the neighbourhood of zero when there are only pure imaginary characteristic exponents and there are no resonances up to the fourth-order inclusive, and, moreover, is of independent interest. All the conclusions will hold when there are additional equations in the variable ξ having the same meaning as in (1.7).

When $\mu = 0$, system (2.1) only allows conditionally periodic motions

$$\eta_s = \sqrt{r_s}e^{i\theta_s}, \quad r_s^{\cdot} = 0, \quad \theta_s^{\cdot} = \omega_s + 2\sum_{j=1}^{n} C_{sj}r_j \quad (s = 1,\ldots,n)$$

Among these motions there are $2\pi k$-periodic motions corresponding to a denumerable set of points with respect to the initial values of r

$$k(\omega_s + 2\sum_{j=1}^{n} C_{sj}r_j^0) = k_s; \quad k, |k_s| \in N$$

In the neighbourhood of the chosen periodic motion we put

$$\eta_s = \sqrt{r_s^0}e^{i\theta_s}(1 + x_s), \quad \bar{\eta}_s = \sqrt{r_s^0}e^{i\theta_s}(1 + \bar{x}_s)$$

The equations for x_s, $\bar{x}_s$ then take the form

$$x_s^{\cdot} = 2i\sum_{j=1}^{n} C_{sj}r_j^0(x_j + \bar{x}_j) + \ldots \quad (s = 1,\ldots,n) \tag{2.2}$$

where the omitted terms are of order not less than the first in $\mathbf{x}$, $\bar{\mathbf{x}}$ or depend on μ and are $2\pi k$-periodic functions of t. In real variables $\mathbf{p}$, $\mathbf{q}$ ($\mathbf{x} = \mathbf{p} + i\mathbf{q}$), system (2.2) takes the form

$$p_s^{\cdot} = 0 + \ldots, \quad q_s^{\cdot} = 2\sum_{j=1}^{n} C_{sj}r_j^0 p_j + \ldots \tag{2.3}$$

Since the set of fixed points of the automorphism for (2.3) coincides with the hyperplane $\mathbf{q} = 0$, the sufficient condition for $2\pi k$-periodic solutions, close to the generating solution, to exist in (2.1) is det $\|C_{sj}r_j^0\| \neq 0$. In fact, in this case the written part of system (2.3) is a special case of system (1.7).

The generating system also has other periodic solutions. In fact, when $\mu = 0$ in (2.1) the hyperplanes $\eta_s = \bar{\eta}_s = 0$ are integral. Hence, if $\eta_{\nu+1} = \ldots \eta_n = 0$ in the generating solution, the sufficient conditions for a $2\pi k$-periodic solution to exist in (2.1) are

$$\omega_\alpha + 2\sum_{j=1}^{\nu} C_{\alpha j}r_j^0 = \frac{k_\alpha}{k}, \quad \omega_\beta + 2\sum_{j=1}^{\nu} C_{\beta j}r_j^0 \neq \frac{l}{k}, \quad \det\left\|C_{\alpha j}r_j^0\right\|_1^\nu \neq 0 \tag{2.4}$$
$$k \in N; \quad k_\alpha, l \in Z \quad (\alpha = 1,\ldots,\nu; \ \beta = \nu + 1,\ldots,n)$$

Theorem 3. When (2.4) is satisfied, system (2.1) has a $2\pi k$-periodic solution for sufficiently small μ, identical with the generating solution when $\mu = 0$.

Corollary. When $C_{ss} \neq 0$ and there are no resonances up to the fourth-order inclusive, Lyapunov families of periodic motions exist for almost all initial conditions with the exception of a denumerable set of points in r_s^0.

In fact, in the autonomous system (2.1) when $C_{11} \neq 0$ any solution of the generating system in the hyperplane $\eta_2 = \ldots \eta_n = 0$ is periodic if $\omega_1^* = \omega_1 + 2C_{11}r_1^0 \neq 0$, which is satisfied for small r_1^0. The condition $l(\omega_\beta + 2C_{\beta 1}r_j^0) \neq k_1\omega_1^*$; k_1, $l \in \mathbf{Z}$ distinguishes a denumerable set of points r_1^0 for which the periodic motions are not continued with respect to the small parameter μ at least according to the theory from Section 1.

3. THE FOURTH-ORDER RESONANCE $4\omega = N$, $N \in \mathbf{N}$

For simplicity we will consider the case of two variables $l = n = 1$ in (1.1). It is easy to generalize the results to the case of arbitrary numbers l and n ($l \geqslant n$) though it leads to more lengthy calculations.

We can write the system in complex-conjugate variables η, $\bar{\eta}$ in the form [1]

$$\eta^{\cdot} = i\left[\omega + C_{1,0}\eta\bar{\eta} + C_{-1,1}(\eta\bar{\eta})^{-1}(\bar{\eta}e^{i\omega t})^4\right]\eta + \mu H(\mu, \eta, \bar{\eta}, t) \tag{3.1}$$

where the function H is 2π-periodic in t, and $C_{1,0}$, $C_{-1,1}$ are real constants.

We make the replacement $\eta = ze^{i\omega t}$ and $\bar{\eta} = \bar{z}e^{-i\omega t}$. We then have

$$z^{\cdot} = i(C_{1,0}z^2\bar{z} + C_{-1,1}\bar{z}^3) + \ldots \tag{3.2}$$

where the unwritten terms are of the order of μ and are 8π-periodic functions of t (we assume that $N = 1$ below). It is convenient to analyse the generating system obtained from (3.2) when $\mu = 0$ in polar coordinates r, θ: $z + \sqrt{(r)}e^{i\theta_1}$, $\theta = 4\theta_1$. We have

$$r^{\cdot} = 2C_{-1,1}r^2\sin\theta, \quad \theta^{\cdot} = 4(C_{1,0} + C_{-1,1}\cos\theta)r$$

Applying the extension [1] of the Heinbockel–Struble theorem [5] to this system we can conclude that when $|C_{1,0}| > |C_{-1,1}|$ all the motions are periodic. We obtain motions along ellipses

$$r(\theta)\frac{\lambda^*}{4C_{1,0}(1+\varepsilon\cos\theta)}, \quad \theta^{\cdot} = \lambda^* = 4r_0(C_{1,0} + C_{-1,1}\cos\theta_0), \quad \varepsilon = \frac{C_{-1,1}}{C_{1,0}} \tag{3.3}$$

(r_0, θ_0 are the initial values of the variables r, θ), where, without loss of generality, we can assume $\theta_0 = 0$. Note that the ellipses intersect the fixed set ($\sin\theta = 0$) at two points.

Among the motions (3.3) we distinguish $8\pi k^*$-periodic motions ($k^* \in \mathbf{N}$) with respect to the variable t, for which $\lambda^* k^* = s \in \mathbf{Z}$. The "amplitudes" of these motions can be found from the relation

$$4r_0(C_{1,0} + C_{-1,1}) = s/k^*$$

If $k^* = 1$, then for any $s \in \mathbf{Z}$ we have motions that are 8π-periodic with respect to t for which the "amplitudes" can be as large as desired (as $|s|$ increases), beginning from the value

$$4r_0^{\min} = \left|C_{1,0} + C_{-1,1}\right|^{-1}$$

Hence, oscillations with the resonance frequency ω occur outside a bounded region contained inside the ellipse (3.3) with $r_0 = r_0^{\min}$. As regards subharmonic oscillations ($k^* = 2, 3, \ldots$), their amplitudes may be as small as desired as k^* increases.

We will investigate under what conditions the periodic motions (3.3) can be continued with respect to the small parameter μ. To do this we put $z = \sqrt{(r)}(\theta)\exp(i\theta_1)(1 + x)$ in (3.2). Then, in the real variables p, q ($x = p + iq$) we obtain

$$\begin{aligned} dp/d\theta &= 2\varepsilon C_{-1,1}(p\sin\theta - q\cos\theta)r^2(\theta)/\lambda^* + \ldots \\ dp/d\theta &= 2C_{-1,0}[(1 + \varepsilon\cos\theta)p + \varepsilon q\sin\theta]r^2(\theta)/\lambda^* + \ldots \end{aligned} \tag{3.4}$$

According to Theorem 1 the absence among the characteristic exponents $\pm\kappa$ of this system of exponents equal to $iN/(4|s|)$, $N \in \mathbf{N}$ guarantees the continuation of the $8\pi k^*$-periodic motion with respect to μ. One can determine κ by constructing in $\theta \in [0, 8\pi|s|]$ one partial solution with initial conditions, for example, $p = 0$ and $q = 1$. However, the problem will be solved here using integrals of the generating system.

The generating system—(3.2) with $\mu = 0$—has the energy integral

$$V = 2C_{1,0}(z\bar{z})^2 + C_{-1,1}(z^4 + z^{-4}) = \text{const}$$

In the neighbourhood of the periodic motion (3.3) the integral can be written in the form

$$[(C_{1,0} + C_{-1,1}\cos\theta)p - C_{-1,1}q\sin\theta)r^2(\theta) + \ldots = h(\text{const}) \tag{3.5}$$

where terms higher than the first order in p and q are omitted. Integral (3.5) enables us to make the new variable h instead of the variable p in (3.4). We obtain the system

$$dh/d\theta = + \ldots, \quad dq/d\theta = 2[h + 2C_{1,0}\varepsilon q \sin\theta]/\lambda^* + \ldots$$

with automorphism $\theta \to -\theta, h \to h, q \to -q$. It is obvious that this system satisfies all the conditions of Theorem 1.

Theorem 4. Suppose that the condition $|C_{1,0}| > |C_{-1,1}|$ is satisfied in system (3.1). Then, for sufficiently small μ, system (3.1) has a denumerable set of $8\pi k^*$-periodic motions close to the motions (3.3). The "amplitude" r_0 of the motions with the resonance frequency will then be a multiple of $r_0^{\min}$.

Note. Taking into account the fact that when $|C_{1,0}| \leq |C_{-1,1}|$ the motions of the generating system are not periodic, we can assert that all the periodic motions that are $8\pi k^*$-periodic in t can be continued with respect to the small parameter.

4. SECOND-ORDER RESONANCE $2\omega = N, N \in \mathbf{N}$

We will consider the problem of the periodic motions of the following system

$$\eta^\cdot = i\omega\eta + i\left[C_{1,0}\eta\bar\eta + C_{-1,2}\bar\eta^3 e^{-4i\omega t} + C_{2,1}\eta^3 e^{2i\omega t} + C_{0,-1}\eta\bar\eta^2 e^{-2i\omega t}\right] + \mu H(\mu, \eta, \bar\eta, t) \qquad (4.1)$$

(the complex-conjugate equation is omitted) with a function H which is 2π-periodic in t and with an automorphism $(t, \eta, \bar\eta) \to (t, \bar\eta, \eta)$, where C_{sj} are real constants. We will make the replacement $z = \eta \exp(-i\omega t)$. The right-hand sides of the system obtained will then be 4π-periodic in t (for brevity we assume $N = 1$) and in polar coordinates r, $\theta(z = \sqrt{(r)}\exp(i\theta_1)$, $\theta = 2\theta_1)$ for $\mu = 0$ we have

$$\begin{gathered} r^\cdot = 2(C_- \sin\theta + C_{-1,2}\sin 2\theta)r^2, \quad \theta^\cdot = 2\Delta(\theta)r \\ \Delta(\theta) = C_{1,0} + C_+\cos\theta + C_{-1,2}\cos 2\theta, \quad C_\pm = C_{0,1} \pm C_{2,1} \end{gathered} \qquad (4.2)$$

The motions of this system will be periodic [1] if $\theta^\cdot \neq 0$ when $r \neq 0$ and for any θ. Hence, the necessary and sufficient conditions for periodicity are

$$D = C_+^2 - 8C_{-1,2}C_* \geq 0, \quad ||C_+| - \sqrt{D}| < 4\sqrt{C_{-1,2}}, \quad C_* = C_{1,0} - C_{-1,2} \qquad (4.3)$$

In this case the relationship $r(\theta)$ is found from the equation

$$\frac{dr}{r} = \frac{C_- \sin\theta + C_{-1,2}\sin 2\theta}{\Delta(\theta)}d\theta = f(\theta)d\theta$$

Hence

$$r(\theta) = r_0 \exp\int_0^\theta f(\theta), \quad \int_0^{2\pi} f(\theta)d\theta = 0$$

We obtain the relationship $\theta(t)$ by integrating the relation

$$r_0 dt = f_1^{-1}(\theta)d\theta, \quad f_1(\theta) = 2\Delta(\theta)\exp\int_0^\theta f(\theta)d\theta$$

In view of the 2π-periodicity of the function $f_1(\theta)$ we obtain

$$r_0 t = [g_1\theta + g_2(\theta)], \quad g_1 = \text{const}, \quad g_2(\theta + 2\pi) = g_2(\theta)$$

Hence it follows that when the condition $r_0 k^* = g_1 s (k^* \in \mathbf{N}, s \in \mathbf{Z})$ is satisfied the motions will be 4π-periodic in t. Hence, as in the case of fourth-order resonance, the amplitude of the vibrations with resonance frequency ($k^* = 1$) is a multiple of $r_0^{\min} = g_1$, while the subharmonic vibrations can have as small an amplitude as desired.

Now, as in the case of fourth-order resonance, we put $z = \sqrt{(r(\theta))} \exp(i\theta_1) \times (1+x), x = p + iq$. We then obtain the system

$$dp/d\theta = [C_{-1,2}(p\sin 2\theta + 2q\cos 2\theta) + C_+(p\sin\theta - q\cos\theta)]/\Delta(\theta) + \ldots$$
$$dq/d\theta = [C_{1,0}p + C_{-1,2}(p\cos 2\theta - 2q\sin 2\theta) + C_+ p\cos\theta - C_- q\sin\theta)]\Delta(\theta) + \ldots \tag{4.4}$$

The unwritten terms are of order higher than the first of p and q or depend on μ and are $4\pi|s|$-periodic functions of θ ($4\pi k^*$-periodic functions of t).

Theorem 5. Suppose conditions (4.3) are satisfied in system (4.1). Then, for fairly small μ Eq. (4.1) has a denumerable set of motions, $4\pi k^*$-periodic in t ($k^* \in \mathbf{N}$), if the characteristic exponents of system (4.4) are not equal to $\pm iN/(2|s|)$, $N \in \mathbf{N}$.

Note. When $C_+ = 2C_-$ the generating system has an energy integral and its periodic motions are continued with respect to the small parameter.

5. 1:3 RESONANCE

Omitting the associated non-resonant subsystem, we will consider the problem of the periodic motions of the system

$$\eta_1^{\cdot} = i(\omega_1 + A_{11}|\eta_1|^2 + A_{12}|\eta_2|^2)\eta_1 + iB_2\bar{\eta}_2^3 + \mu H_1(\mu, \bar{\eta}, \eta)$$
$$\eta_2^{\cdot} = i(-\omega_2 + A_{21}|\eta_1|^2 + A_{22}|\eta_2|^2)\eta_2 + iB_2\bar{\eta}_1\eta_2 + \mu H_2(\mu, \bar{\eta}, \eta) \tag{5.1}$$

where A_{jk}, B_j, ω_j are real constants, $\omega_j > 0$, $\omega_1 = 3\omega_2$, and the complex-conjugate group of equations is omitted.

The generating system, obtained from (5.1) when $\mu = 0$, always has a periodic solution in which

$$\eta_1^{\cdot} = i(\omega_1 + A_{11}|\eta_1|^2)\eta_1, \quad \eta_2 = 0 \tag{5.2}$$

By Theorem 3, system (5.1) for small μ has periodic solutions close to (5.2). It also follows from Theorem 3 that periodic motions exist close to the solutions

$$\eta_1 = 0, \quad \eta_2^{\cdot} = i(-\omega_2 + A_{22}|\eta_2|^2)\eta_2$$

if $B_1 = 0$.

It turns out that other "resonant" periodic motions are also possible in system (5.1). To establish this fact we convert the generating system to polar coordinates r, θ: $\eta_j = \sqrt{(r_j)} \exp(i\theta_j)$, $\bar{\eta}_j = \sqrt{(r_j)} \exp(-i\theta_j)$ ($j = 1, 2$). We obtain

$$r_j^{\cdot} = 2B_j \sin\theta r_1^{1/2} r_2^{3/2} \quad (j = 1, 2)$$
$$\theta_1^{\cdot} = \omega_1 + A_{11}r_1 + A_{12}r_2 + B_1 r_1^{-1/2} r_2^{3/2}\cos\theta \tag{5.3}$$
$$\theta_2^{\cdot} = -\omega_2 + A_{21}r_1 + A_{22}r_2 + B_2 r_1^{1/2} r_2^{1/2}\cos\theta, \quad \theta = \theta_1 + 3\theta_2$$

where the last two equations reduce to a single equation for the variable

$$\theta^{\cdot} = A_1 r_1 + A_2 r_2 + (B_1 r_1^{-1/2} r_2^{3/2} + 3B_2 r_2^{1/2} r_2^{1/2})\cos\theta, \quad A_j = A_{1,j} + 3A_{2j}.$$

A qualitative investigation of the system of equations obtained for r_1, r_2, θ [6] enabled, in particular, all cases of the existence of periodic motions to be established. Suppose $B_1B_2 > 0$, for example, $B_{1,2} > 0$. Then (5.3) has a particular solution described by the equations

$$r^{\cdot} = 2Br^2\sin\theta, \quad \theta^{\cdot} = (A + 4B\cos\theta), \quad r_j = B_j r \quad (j = 1, 2) \tag{5.4}$$

$$A = A_1B_1 + A_2B_2, \quad B = B_1^{1/2}B_2^{3/2}$$

When the condition $|A| > 4|B|$ is satisfied all the solutions of system (5.4) will be periodic motions along ellipses

$$r(\theta) = r_0\sqrt{\frac{1+\varepsilon}{1+\varepsilon\cos\theta}}, \quad \varepsilon = 4\frac{B}{A}$$

where r_0 is the value of r when $\theta = 0$. In $\eta, \bar{\eta}$ variables the motion, generally speaking, will be conditionally periodic, and

$$\theta^{\cdot}_1 = \omega_1 + (a_1 + B\cos\theta)r, \quad \theta^{\cdot}_2 = -\omega_2 + (a_2 + B\cos\theta)r, \quad a_j = \sum_{k=1}^{2} A_{jk}B_k$$

and the dependence of θ_j on 0 is given by the relations

$$\theta_{1,2} = \pm\frac{\omega_{1,2}}{Ar_0\sqrt{1+\varepsilon}}[\alpha^*\theta + f_1(\theta)] + \frac{a_{1,2}}{A}[\beta^*\theta + f_2(\theta)] + \frac{B}{A}[\gamma^*\theta + f_3(\theta)] \tag{5.5}$$

Here $f_j(\theta)$ is a 2π-periodic function of θ, and α^*, β^*, γ^* are the average values over a period of the functions

$$\alpha^* = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\sqrt{1+\varepsilon\cos\theta}}, \quad \beta^* = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}, \quad \gamma^* = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos d\theta}{1+\varepsilon\cos\theta}$$

Hence, the solution will be periodic if the following condition is satisfied

$$\omega_1^* k = \omega_2^* l, \quad \omega_{1,2}^* = \pm\frac{\omega_{1,2}\alpha^*}{Ar_0\sqrt{1+\varepsilon}} + \frac{a_{1,2}\beta^* + B\gamma^*}{A}$$

where k and l are integers, excluding zero, and the period with respect to θ is equal to $|2\pi k/(\omega^*_1 l)|$. Note also that the periodicity condition is satisfied irrespective of the initial value of r_0 of the generating solution if $k + 3l = 0$.

To apply Theorems 1 and 2 to the established elliptic generating solutions we change to the variables p_j and q_j in (5.1) using the formulae

$$\eta_j = \sqrt{|B_j| r(\theta)} e^{i\theta_j}(1 + x_j), \quad x_j = p_j + iq_j \quad (j = 1,2)$$

where the relationship $\theta_j(\theta)$ is given by (5.5). We obtain

$$\begin{aligned}
\frac{dp_1}{d\theta} &= \varepsilon\frac{(3p_2 - p_1)\sin\theta + (q_1 + 3q_2)\cos\theta}{4(1+\varepsilon\cos\theta)} + \ldots \\
\frac{dq_1}{d\theta} &= \varepsilon\frac{2(A_{11}B_1p_1 + A_{12}B_2p_2)}{A(1+\varepsilon\cos\theta)} + \varepsilon\frac{(3p_2 - p_1)\cos\theta - (q_1 + 3q_2)\sin\theta}{4(1+\varepsilon\cos\theta)} + \ldots \\
\frac{dp_2}{d\theta} &= \varepsilon\frac{(p_1 - p_2)\sin\theta - (q_1 + q_2)\cos\theta}{4(1+\varepsilon\cos\theta)} + \ldots \\
\frac{dp_2}{d\theta} &= \frac{2(A_{21}B_1p_1 + A_{22}B_2p_2)}{A(1+\varepsilon\cos\theta)} + \varepsilon\frac{(p_1 - p_2)\cos\theta + (q_1 - q_2)\sin\theta}{4(1+\varepsilon\cos\theta)} + \ldots
\end{aligned} \tag{5.6}$$

The unwritten terms are of order higher than the first in **p**, **q** or depend on μ.

It can be seen that when $\varepsilon = 0$ the part in (5.6) that is linear in **p**, **q** and free from μ is independent of θ, the characteristic equation has two pairs of zero roots, each from one group of solutions, and the automorphism satisfies Theorem 1. Consequently, when $\varepsilon = 0$ system (5.1) has a denumerable set of periodic motions close to circular for sufficiently small μ.

To determine the characteristic indices when $\varepsilon \neq 0$ we will use the integrals [6] of the generating system

$$V = B_1\eta_2\bar{\eta}_2 - B_1\eta_1\bar{\eta}_1$$

$$W = A_1B_2|\eta_1|^4 + A_2B_1|\eta_2|^4 + 2B_1B_2(\eta_1\eta_2^3 + \bar{\eta}_1\bar{\eta}_2^3)$$

In the neighbourhood of the elliptic generating solution the integrals take the form

$$(p_1 - p_2)r(\theta) + \ldots = h_1$$
$$\left\{A_1B_1p_1 + A_2B_2p_2 + B\left[(p_1+3p_2)\cos\theta - (q_1+3q_2)\sin\theta\right]\right\}r^2(\theta) + \ldots = h_2$$

where h_1 and h_2 are arbitrary constants. It can be shown that these integrals are solvable with respect to p_1 and p_2 provided the condition $|A| > 4|B|$ is satisfied. Hence, in system (5.6) we can take the new variables h_1 and h_2 instead of the variables p_1 and p_2. As a result it turns out that the linear system has two pairs of zero characteristic exponents, each from one group of solutions, and we obtain, apart from terms that are linear and free from μ

$$h_1^{\cdot} = 0 + \ldots, \quad h_2^{\cdot} = 0 + \ldots$$

In the new variables h_j, q_j ($j = 1, 2$) the automorphism has the form $t \to -t, h \to h, q \to -q$ and all the conditions of Theorem 1 are satisfied.

Theorem 6. If $|A| > 4|B|$, in system (5.1) for sufficiently small μ there will always be a denumerable set of "resonant" periodic motions close to elliptic and coincident with them when $\mu = 0$.

Note. The presence of two independent integrals enables all the periodic solutions established in the generating system [6] to be extended with respect to the small parameter. A similar situation occurs for an arbitrary fourth-order m-frequency resonance because in this case the generating system has m independent integrals [7].

6. PERIODIC MOTIONS IN THE UNRESTRICTED THREE-BODY PROBLEM

We will consider the plane unrestricted three-body problem—the problem of the motion of three mass points P_0, P_1 and P_2 with masses M_0, M_1 and M_2, which mutually attract one another in accordance with Newton's law, and which always move in the same fixed plane. The Routh function of the problem was obtained in [8] and the following were chosen as the positional coordinates: r—the square root of the polar moment of inertia, y—the natural logarithm of the ratio of the two sides P_0P_1 and P_0P_2 of the triangle $P_0P_1P_2$, and ψ—the angle between these sides. The cyclic variable Φ is the angle measured from the straight line P_0P_1 from a certain fixed straight line in the plane of the triangle $P_0P_1P_2$. We have

$$R = r^{\cdot 2} + r^2F_2 + F_1 - \frac{\beta^2}{4r^2} + \frac{1}{r}F_0, \quad F_2 = \mu S^{-2}(y^{\cdot 2} + \psi^{\cdot 2}) \tag{6.1}$$
$$F_1 = -\beta S^{-1}\left\{\mu_3(\psi^{\cdot}\cos\psi + y^{\cdot}\sin\psi) - (\mu_2+\mu_3)e^y\psi^{\cdot}\right\}$$
$$F_0 = fM\sqrt{S/2}\left\{\mu_1 e^{y/2} + \mu_2 e^{-y/2} + \mu_3(e^y + e^{-y} - 2\cos\psi)^{-1/2}\right\}$$
$$S = \mu_1 e^{-y} + \mu_2 e^y + \mu_3(e^y + e^{-y} - 2\cos\psi)$$
$$\mu_{i+j} = M_iM_j/M^2 \quad (i,j = 0,1,2;\ i \neq j), \quad \mu = \mu_0\mu_1 + \mu_0\mu_2 + \mu_1\mu_2$$

where M is the mass of the whole system, β is the cyclic constant, and f is the gravitational constant.

The distinguishing features of this description of the problem are as follows. The new parameters of the problem are the dimensionless products μ_{i+j} of the masses of the bodies P_i and P_j, and these parameters reflect the Newtonian interaction between these bodies. The equation of motion for the variable r

$$2r^{\cdot\cdot} = 2rF_2 + \frac{\beta^2}{2r^3} - \frac{1}{r^2}F_0 \tag{6.2}$$

in fact is identical with a fundamental relation in celestial mechanics, namely, the Lagrange–Jacobi equation, and expresses in differential form the fact that the mechanical energy is conserved. Finally, the problem is described by Routh's equations, which at the same time are reversible to the replacement $(r, y, \psi) \to (r, y, -\psi)$.

The introduction of new parameters enables us, as the limiting version when $\mu_3 = 0$, to obtain the problem

$$\frac{d}{dt}\left(\frac{r^2}{S^2}y^{\cdot}\right)+\frac{r^2}{S^3}(y^{\cdot 2}+\psi^{\cdot 2})\frac{\partial S}{\partial y}-\frac{\beta}{S^2}\psi^{\cdot}+\frac{1}{r}\frac{fM}{2\sqrt{2S}}(e^{-3y/2}-e^{3y/2})=0$$
$$\frac{d}{dt}\left(\frac{r^2}{S^2}\psi^{\cdot}\right)+\frac{\beta y^{\cdot}}{S^2}=0 \qquad (6.3)$$

which corresponds to the case when there is no interaction between the bodies P_1 and P_2; these bodies move in a rotating system of coordinates only under the influence of the body P_0.

Equations (6.2) and (6.3) allow steady motion in which $r, y, \psi^{\cdot}$ take constant values. In such motion the bodies P_1 and P_2 rotate with constant angular velocities $\psi^{\cdot}$ and $\Phi^{\cdot} + \psi^{\cdot}$ around the centre of mass of the bodies P_0, P_1 and P_2, and of course, around P_0. The angular velocities $\Phi^{\cdot}$ and $\psi^{\cdot}$ are then related by the equation

$$2r^2\Phi^{\cdot} = \beta - \mu_2 r^2\psi^{\cdot} \qquad (6.4)$$

and the motion occurs along circles.

All the steady solutions of system (6.2), (6.3) are found from the equation

$$x^2(-\mu_1^2e^{-y/2}+\mu_2^2e^{y/2})-\beta x(\mu_1e^{y/2}+\mu_1e^{-y/2})+\frac{\beta^2}{4}(e^{-3y/2}-e^{3y/2}) = 0 \qquad (6.5)$$

where $x = r^2\psi^{\cdot}/S$.

We will investigate different possible cases.

1. Suppose the coefficient of x^2 is zero. The ratio of the distances is then $r_2/r_1 = (\mu_1/\mu_2)^2$. If $\beta = 0$, any value of x is a solution of Eq. (6.5). The ratios of the angular velocities and the periods in this case are

$$\frac{\omega_2}{\omega_1}=\frac{\Phi^{\cdot}+\psi^{\cdot}}{\Phi^{\cdot}}=-\left(\frac{\mu_2}{\mu_1}\right)^3,\quad \left(\frac{T_1}{T_2}\right)^2=\left(\frac{r_1}{r_2}\right)^3$$

Hence, in this steady motion, the bodies P_1 and P_2 rotate in opposite directions, and Kepler's law is satisfied: the square of the ratio of the periods is equal to the cube of the ratio of the radii.

We will now assume that $\beta \neq 0$. Then, we obtain the following unique solution of Eq. (6.5) for x

$$x=\frac{\beta}{4}\frac{\mu_2^3-\mu_1^3}{\mu_1^2-\mu_2^2}$$

Hence, two bodies of equal mass ($\mu_1 = \mu_2$) move with the same angular velocity along the same circle, since for this system $\psi^{\cdot} = 0$. Suppose ($\mu_1 \neq \mu_2$). Then

$$\frac{\psi^{\cdot}}{\Phi^{\cdot}}=\frac{\mu_*^6-1}{2\mu_*^3[2+\mu_*(1-\mu_*^3)]}=f(\mu_*),\quad \mu_*=\frac{\mu_2}{\mu_1}=\frac{M_2}{M_1}$$

If $\mu_* \ll 1$, we have $\omega_2/\omega_1 < 0$, and the rotations occur in opposite directions. Here the body P_2 lies on a circle of larger radius than the body P_2. If now $\mu_* \to \infty$ we have $\omega_2/\omega_1 \to -1/2$, and the bodies P_2 and P_1 move close to resonance.

2. Suppose the coefficient of x^2 in (6.5) is non-zero. Then x are the roots of the quadratic equation and have the form

$$x=\frac{\mu_1e^{y/2}+\mu_2e^{-y/2}\pm(\mu_1e^{-y}+\mu_2e^{y})}{2(-\mu_1^2e^{-y/2}+\mu_2^2e^{y/2})}\beta$$

We obtain two families of steady motions: for each value of y and for any μ_1, μ_2, which do not make the denominator vanish, there are two values of the angular velocity $\psi^{\cdot}$ of the motion of P_2 with respect to P_1. We calculate

$$\frac{\psi^{\cdot}}{\Phi^{\cdot}}=-\frac{e^{y/2}+\mu_*e^{-y/2}\pm(e^{-y}+\mu_*e^{y})}{e^{-y/2}+\mu_*e^{3y/2}\pm\mu_*(1+\mu_*e^{2y})}(e^{-y}+\mu_*e^{y})=f_1(\mu_*)$$

When $\mu_* \to 0$ we have $f_1(\mu_* \to -(1 \pm e^{-3y/2})$, and when $\mu_* \to \infty$ the function $f_1(\mu_*)$ approaches $\pm(1 + e^{-3y/2})$. Hence it follows that when one of the masses M_1, M_2 is much greater than the other mass, rotations of the bodies are possible both in the same direction and in opposite directions. In the case when the bodies rotate with the same angular velocity we have $r_1 = r_2$.

Suppose now that μ_* is a finite quantity, possibly fairly large. We remove the body P_2 to infinity. Then $y \to \infty$ and $\psi^{\cdot}/\Phi^{\cdot} \to -1$. We obtain a version of the problem which describe the motion of the Moon P_1 with angular velocity $\Phi^{\cdot}$ around the Earth P_0. In this version P_2 represents the Sun and the angular velocity of rotation of the Earth around the Sun is much less than the angular velocity of rotation of the Moon around the Earth.

We will now set up equations in variations for system (6.2), (6.3) in the neighbourhood of the steady motions considered. We will not write these fairly length equations explicitly and will merely denote their structure

$$\delta\psi^{\cdot\cdot} = -\beta/r^{-2}\delta y$$

$$\delta r^{\cdot\cdot} = a\delta\psi^{\cdot} + a_{11}\delta r + a_{12}\delta y, \quad \delta y^{\cdot\cdot} = b\delta\psi^{\cdot} + a_{21}\delta r + a_{22}\delta y$$

where a, b and a_{ij} are certain constant coefficients which depend on the parameters μ_1, μ_2 and the steady motion considered.

It can be seen that this system has a pair of zero characteristic exponents with one group of solutions, while the automorphism of the system satisfies Theorem 1. The remaining characteristic exponents are defined as the roots of a biquadratic equation. These numbers depend on the parameters μ_1 and μ_2 and will be critical in the sense that it is impossible to extend the periodic solution of system (6.2), (6.3) with respect to μ_3 except for a denumerable set of values of μ_1 and μ_2.

Theorem 7. For a sufficiently small value of the parameter μ_3 the plane unrestricted three-body problem has "circular" periodic solutions in which the bodies P_1 and P_2 rotate around P_0 with constant angular velocity (to an accuracy of the order of μ_3) along curves close to circles. The motions of the bodies P_1 and P_2 can then be both in the same and in opposite directions.

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